Quasiclassical surface of section perturbation theory

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Perturbation theory, the quasiclassical approximation, and the quantum surface of section method are combined. This gives a new solution of the long-standing problem of quantizing the resonances generically appearing in classical perturbation theory. Our method is restricted to two dimensions. In that case, however, the results are simpler, more explicit, and more easily expressed visually than the results of earlier techniques. The method involves expanding the ''phase'' of the wave function in powers of the *square root* of the small parameter. It gives explicit WKB-like wave functions and energies even for certain systems which classically show hard chaos. It also gives certain classes of states in some nonintegrable systems. The relationship of the method to earlier techniques is discussed. [S1063-651X(99)13502-5]

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I. INTRODUCTION

Quasiclassical approximations to quantum theory have been valuable not only for numerical purposes but because much insight is thereby gained. WKB theory for the one-dimensional case is textbook material. For more general integrable systems, Einstein-Brillouin-Keller (EBK) theory is used. The Gutzwiller trace formula (GTF) [1], is the best known technique applicable to hard chaos systems. A completely quasiclassical method is not available for mixed chaotic systems.

In this paper, we study the quasiclassical quantization of *perturbed* two-dimensional *integrable* systems by a new technique [2]. The method also applies to *special classes of states* of certain nonintegrable systems. Our technique is formulated on a *surface of section*, by means of the Bogomolny operator T [3]. This effectively reduces the problem to one dimension and we find directly, by quasiclassical, WKB-like methods, the eigenfunctions and eigenenergies of the system.

There is a long history of interest in this subject. The work of Poincaré on classical perturbation theory eventually culminated in the KAM theory [1,4,5] of Kolmogorov, Arnol'd, and Moser and the ideas of chaos theory. Several methods, such as that of Birkhoff-Gustavson [6] and the perturbed Berry-Tabor trace formula [7], have been used to quantize such perturbed systems quasiclassically. We will give a more detailed comparison at the end.

A major feature of perturbed classically integrable systems is that the long-time behavior typically shows phase-space structure at all scales of action. This is also true of chaotic systems and perturbed classically integrable systems have much in common with systems displaying mixed chaos. Such phenomena, and their relation to quantum ideas in particular examples, have motivated a considerable number of recent publications.

Thus we are interested in the case where *two* small parameters are present. The first parameter, which is classical, is denoted ϵ. It gives the scale of the difference between an exactly solvable, integrable case, and the ''perturbed'' case of interest. The second parameter is the dimensionless Planck's constant, $\hbar$, which gives the scale of the leading-order quantum or wave effects.

The *relationship* between these parameters is crucial, and the theory is exceedingly rich as a result. Although this has certainly been understood for some time, we have not found a very clear discussion in the context of the formal perturbation-quasiclassical approaches used up to now.

The relationship comes from the well known fact, made manifest by Feynman's path-integral formulation, that quantum effects smear classical phase-space structures over areas of size $h=2\pi\hbar$. We choose to compare Planck's constant with the area of the leading phase-space structure, call it $\mathcal{S}$. The *dimensionless* Planck's constant is $h/\mathcal{S}$, which we continue to call h, i.e., we choose $\mathcal{S}$ as the unit of action. For example, in a nearly circular billiard of radius R, $\mathcal{S}$ could be $\mathcal{S}=pR=\hbar kR$, so $h/\mathcal{S}=2\pi/kR=\lambda/R$. Here p is the momentum, $k=p/\hbar$ the wave number, and λ the wavelength of the particle in the billiard. If the dimensionless Planck is of order unity, only the gross features of the classical system are reflected in the quantum properties. But for small ϵ, or in the presence of chaos, there are classical structures on small scales, e.g., on the scale of ϵ^β, where β ranges from zero to infinity. If ϵ^M/h is of order unity, it means that quantum systems do not reflect the classical structure at the levels ϵ^β with $\beta>M$ but *are* sensitive to structure with $\beta\leqslant M$. Typically, the smallest nonvanishing β of interest is $\beta=\frac{1}{2}$. This means that there can be striking effects if $\sqrt{\epsilon}>\hbar$, even if, for example, $\epsilon\ll\hbar\ll 1$.

A. Integrable classical systems

An integrable classical system has enough constants of the motion that each classical orbit in $2d$-dimensional phase space lies on a d-dimensional surface in that space. These surfaces turn out to be tori. The d constants of the motion can be taken to be the *action variables*, I_j, $j=1,\ldots,d$, and they label the *invariant tori*. From now on, we take $d=2$. In these canonical variables, the Hamiltonian H_0 depends only on the actions, i.e., $H_0(I_1,I_2)=E$, where E is the energy. The conjugate *angle variables*, θ_1,θ_2, satisfy Hamilton's equations $\dot{\theta}_j=\partial H_0/\partial I_j\equiv\omega_j=\omega_j(I_1,I_2)$. The angle variables fix the point on a given invariant torus. We shall also assume, to avoid certain complications, a principle of ''sufficient nonlinearity,'' which posits that the winding number ω_1/ω_2 is

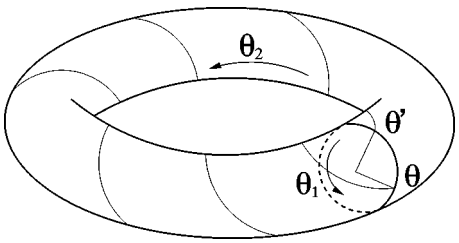

FIG. 1. Schematic of the definition of surface of section. An orbit goes around the hole once and advances by an angle $2\pi\omega_1/\omega_2$ on the SS.

not too constant as I_1 and I_2 are varied at fixed E. We also often use two other constants equivalent to the I's, usually E and the winding number.

B. Poincaré surface of section

A Poincaré surface of section, SS, is a phase-space surface through which all tori of interest pass, once and only once. In action angle variables, or their equivalent, we usually take the surface $\theta_2=0$, E fixed. Figure 1 illustrates such a surface. A given orbit crosses the SS each time θ_2 is a multiple of 2π. Given θ_1' and I_1' when $\theta_2=0$, the dynamics predicts a new $(I,\theta)=\mathcal{T}(I',\theta')$ at the next intersection of the orbit with the SS. (From now on, we call the variables on the SS simply I,θ, without subscripts.) $\mathcal{T}$ is called the *surface of section map* and of course it depends on E. For an integrable system, $I=I'$, $\theta=\theta'+2\pi\omega_1/\omega_2$. For a nonintegrable system, I is not constant.

The surface of section is usually displayed in a graph of I versus θ, for $0\leqslant\theta<2\pi$, and with points $0,2\pi$ identified. The intersection of the invariant torus $I=$const with the surface of section is the horizontal line $I=$const on this graph.

The surface of section map is conveniently given by a *generating function*. This is an action $S(\theta,\theta')$, such that $I=\partial S(\theta,\theta')/\partial\theta$, $I'=-\partial S(\theta,\theta')/\partial\theta'$. In the integrable case,

$$S(\theta,\theta')=S_0(\theta-\theta')=(\theta-\theta')I_1+2\pi I_2, \tag{1}$$

where the actions are regarded as functions of $\theta-\theta'=2\pi\omega_1/\omega_2$ and E. We note that S is just the integral $\int p\,dq$ along the trajectory from one surface of section crossing to the next.

Under perturbation, S becomes

$$S(\theta,\theta')=S_0(\theta-\theta')+\epsilon S_2(\theta,\theta')+\epsilon^2 S_4+\cdots. \tag{2}$$

Another way of describing the perturbation is to give a Hamiltonian

$$H=H_0(I_1,I_2)+\epsilon H_2(I_1,I_2,\theta_1,\theta_2)+\epsilon^2 H_4+\cdots, \tag{3}$$

where the perturbation is periodic in the angles. There is no fundamental difficulty in using perturbation methods to find S_2 given, say, H_2, since the calculation involves only short orbits.

C. Quantum surface of section method

Our theory is expressed in terms of S. Bogomolny [3] has shown (without requiring ϵ small) that the *surface of section transfer operator* or kernel

$$T(\theta,\theta')=\left(\frac{1}{2\pi i\hbar}\frac{\partial^2 S(\theta,\theta')}{\partial\theta\partial\theta'}\right)^{1/2}\exp\left(\frac{i}{\hbar}S(\theta,\theta')\right) \tag{4}$$

can be used to find the energy levels of a system, in quasiclassical approximation, although generally the solution is not very explicit. We mention some of the main results below. Bogomolny's method is by no means restricted to action angle variables or any particular surface of section. There is a large literature [8,9] exploiting and verifying this technique. We call this the *quantum surface of section method.*

We shall find that using Eq. (2) in Eq. (4) and exploiting the small parameter ϵ allows a rather complete and explicit solution of the quantum problem.

D. KAM theory

Although our theory is also applied to non-KAM cases, it naturally reproduces some of the main results of KAM theory, although no attempt is made at rigor. We paraphrase KAM theory as follows. A Hamiltonian $H=H_0+\epsilon H_2$ is assumed, where H_2 is a suitably nice function of I_j,θ_j. The original invariant tori which are *rational*, or equivalently *resonant*, are destroyed, as well as those in their immediate vicinity. The winding number ω_1/ω_2 of such a torus is a rational, p/q. Any orbit on such a torus is *periodic.*

The $\sqrt{\epsilon}$ characteristically appears. Namely, within a width in action, $\sqrt{\epsilon}I_{pq}$ about the rational torus, the neighboring tori, both rational and irrational, are destroyed or modified strongly. Here I_{pq} is a characteristic classical scale of action associated with the pq torus, which vanishes rapidly for large q. We will find an expression for I_{pq}.

The rational tori are dense, but since $\Sigma_{pq}I_{pq}<\infty$, the total volume in which the original invariant tori are destroyed is a small but finite fraction of phase space, proportional to $\sqrt{\epsilon}$. The rest of the tori, the KAM set, remain invariant and are only slightly modified under perturbation. This is the main result of KAM theory.

The destroyed tori are largely replaced by *new* invariant tori with a new topology. Separating these new tori from one another and from the KAM set is a separatrix region which is a chaotic homoclinic tangle. The phase-space scale of the stochastic region is very small [5], namely of order $\exp(-1/\sqrt{\epsilon})$, so our perturbation theory cannot deal with it.

These characteristics are illustrated in the surface of section plot of Fig. 2, which show, for the ''smoothed stadium'' model specified in Sec. II B, the intersections of invariant tori with the surface of section, which we call invariant curves or loops. The largest resonant islands replace the $pq=1,2$ unperturbed torus and its neighbors to a width $\sqrt{\epsilon}$. The width of the 7,16 islands, for example, is much smaller. The ''wave'' on which these islands are riding scales with ϵ, rather than $\sqrt{\epsilon}$. Also shown is an invariant loop which is weakly perturbed, with a winding number corresponding to the golden mean. Several separatrix stochastic regions are also shown.

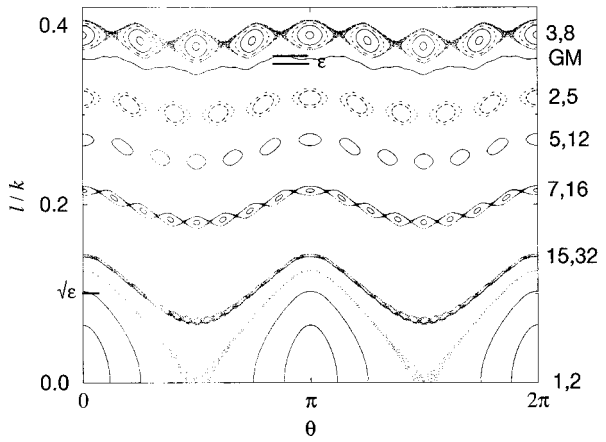

FIG. 2. Invariant loops, the intersection of the surface of section with the invariant tori and separatrices, showing the scale $\sqrt{\epsilon}$ of resonances, the scale ϵ of the ''golden mean'' torus in the KAM set, and, not to scale, secondary resonances. The system is the ''smoothed stadium'' with parameters $\epsilon=0.1$, $\eta=0.19$.

In the process of solving the quantum problem we find simple approximate expressions for the loops of Fig. 2.

The appearance of the square root $\sqrt{\epsilon}$ is generic in the sense that it occurs when the second derivative $S_0''(\theta)$ does not vanish at a value of $\theta=\Theta_{pq}=2\pi p/q$ explained further below. It is easy to construct examples for which this fails. Our method can be generalized to deal with such cases.

II. PREVIEW OF MAIN RESULTS

In many cases, we find explicit formulas for all energy levels and wave functions to leading order in $\hbar$. The equation we solve, approximately, is $\psi=T\psi$, or

$$\psi(\theta)=\int d\theta' T(\theta,\theta';E)\psi(\theta'), \tag{5}$$

where T is the Bogomolny operator. This equation can be solved only if E, on which T depends parametrically, is on the spectrum or more precisely is on a quasiclassical approximation to the spectrum. There generally exists an exact operator or kernel K to which T is a quasiclassical approximation, for which this procedure gives exact answers [9].

For billiards, with the boundary as SS, Eq. (5) is the quasiclassical approximation to the equation of the boundary integral method, and ψ is the normal derivative of the usual wave function $\Psi(\mathbf{r})$ on the boundary. There is a known procedure to find Ψ given ψ. If this procedure is carried out quasiclassically, one finds that Ψ is a generalization of the Born-Oppenheimer or adiabatic approximation, which makes an ansatz for the wave function. We give these results separately [10]. However, ψ contains most of the information desired rather directly, and it is not necessary as a rule to find Ψ. In fact, there are many ways available to represent a complex function of two variables, and ψ is a useful one even in nonperturbative contexts.

These results rely on $\hbar\ll 1$, $\epsilon\ll 1$, and $\epsilon^{M/2}/\hbar\ll 1$, for an appropriate M. The largest possible value of M depends on the problem, but in this paper we shall usually take $M=3$ or 4.

There are also applications to interesting cases where *some*, but not all, states and energies are found, and there is no explicit small parameter ϵ. Rather, for the class of states where the method works, there is an implicit small parameter which appears. The bouncing ball states in the stadium billiard and the whispering gallery states of general smooth convex billiards are cases of this type. These cases are usually connected to an adiabatic approximation and will be discussed in more detail elsewhere.

Our method can be applied to a number of problems which have recently been studied in the literature. We shall list these at the end of this paper.

III. OUTLINE OF THE METHOD

A. Bogomolny integral equation

According to Bogomolny, the energy levels of the system are given in quasiclassical approximation (QCA) [3] by solutions $E=E_a$ of

$$D(E)=\det(1-T(E))=0. \tag{6}$$

This equation is usually approached in one of three ways.

(i) The imaginary part of the logarithmic derivative of D, $d\ln(D)/dE$, can be expanded in traces of powers of T, which yields, for example, the Gutzwiller trace formula (GTF), if all periodic orbits are isolated and unstable [3]. The original derivation of these results was fairly difficult [1], and it had the defect that the resulting sum was mathematically rather ill-defined, since it is not absolutely convergent. The organization of the series by the T operator groups together orbits coming from the same power of T and at least yields a series which either converges or diverges. If T is integrable, this gives the Berry-Tabor result [32], and in the present case, it gives the perturbed Berry-Tabor results mentioned above [7,11,12,32].

(ii) The Fredholm determinant D may be expanded by the rules of Fredholm theory [3,9], giving an absolutely convergent expression, which in quasiclassical approximation is a finite sum. An important improvement uses the unitarity of T to make each term in this sum real. The same traces of powers of T and periodic orbits appear as in the trace formulas, but organized into ''pseudo-orbits'' or ''composite orbits.'' This is the main result of ''resummation'' of the GTF [3,13,14].

(iii) The kernel T may be represented by a discrete matrix, and numerically diagonalized. This method gives very good results [8].

Our seemingly more difficult technique finds wave functions ψ satisfying Eq. (5) which can be done only for $E=E_a$. Our method is tractable only if the orbits, or a sufficiently large subset of orbits, are nearly nonisolated, as is the case for a perturbed integral system. The integral is done in the stationary phase $[S\Phi]$ approximation.

This problem is naturally generalized to

$$T\psi=e^{i\omega}\psi \tag{7}$$

which can be solved for all E, and E is regarded as a parameter. This allows a study of perturbed quantum maps in addition to two-dimensional autonomous systems. The phase ω

will be a function of E and usually it is rather easy to solve the equation $\omega(E)=2\pi n$ which gives the E values such that a solution of Eq. (5) exists.

Using Eqs. (4) and (2), for $\epsilon=0$ a solution of Eq. (5) is $\psi_0=e^{il\theta}$. The $S\Phi$ point is $\theta'=\theta-\Theta_l$, and the action for which this occurs is $I_1=\hbar l$. If Θ_l is near $\Theta_{pq}=2\pi p/q$, for q not too large, this solution will be near a rational torus, which is strongly perturbed. If not, a weaker effect from the perturbation is expected. In general, l will be large, inversely proportional to $\hbar$, but the case $l=0$ is also very interesting. We refer to l as the ''angular momentum.''

We construct a solution of the full problem from functions of the form $\psi=\psi_0\exp(i\alpha f(\theta)/\hbar)$, where $f'\sim 1$ and α determines the rate of variation of the phase. On the assumption, appropriate for the resonant case, that $\epsilon\ll\alpha\ll 1$, it turns out that $\alpha\sim\sqrt{\epsilon}$, while $\alpha\sim\epsilon$ is correct for the nonresonant case. More generally, we will replace αf by a series $bf_1+b^2f_2+\cdots$, where $b=\sqrt{\epsilon}$, in the resonant case, and by a series $\epsilon f_2+\epsilon^2 f_4+\cdots$ in the nonresonant case.

Let integer M be such that $b^{M-1}\gg\hbar$ while $b^{M+1}\ll\hbar$. In *phases* that are expanded in powers of b, we must keep terms up to b^M, even though they are much smaller than other contributions to the phase. This is because such phases give changes of order unity to the wave function as the angle is varied. In this paper we shall usually consider $M=2$, i.e., we shall consider $b^3\ll\hbar$.

In prefactors, we may keep only the leading-order terms and succeed in making only fractionally small errors. In finding the prefactor of the wave function, we shall need to keep terms of order unity, which we may regard as being of the same order as $b^M/\hbar$.

B. Example: Perturbed circle billiard

It is particularly easy to give the actions S_0 and S_2 in the case of nearly integrable billiard systems. The simplest case, which nevertheless has been of considerable interest [15–20], is a particle in a perturbed circular billiard. The radius of the billiard is written $r(\theta)=R_0+\epsilon\Delta R(\theta)$. We may suppose the angular average of ΔR vanishes.

We give illustrative numerical results for two cases. The nearly circular Bunimovich ''stadium'' [16,19,20] has $\Delta R(\theta)=|\sin\theta|-2/\pi$. This has a discontinuous first derivative, invalidating KAM theory, but our theory still works if $\epsilon^2\ll\hbar$. We also use a ''smoothed stadium,'' $\Delta R(\theta)=\sqrt{\sin^2\theta+\eta^2}-C_\eta$, where the constant C_η is chosen to make the angular average vanish. For ϵ sufficiently small compared to η, KAM theory applies. We take $\eta=\sqrt{\epsilon}$, which is large enough for KAM theory to work numerically.

In quantum language, we take units $R_0=1$, $\hbar=1$, particle mass $=1/2$, so $k=\sqrt{E}$ is the dimensionless wave number. We use k and the dimensionless $1/\hbar$ interchangeably. We take the billiard boundary ∂B as SS. The action is

$$S/\hbar=kL(\theta,\theta')=k[L_0(\theta-\theta')+\epsilon L_2(\theta,\theta')+\cdots], \quad (8)$$

where L is the chord length between points θ,θ' on the boundary. We label boundary points by angle instead of the more customary distance around the perimeter. It is easy to find

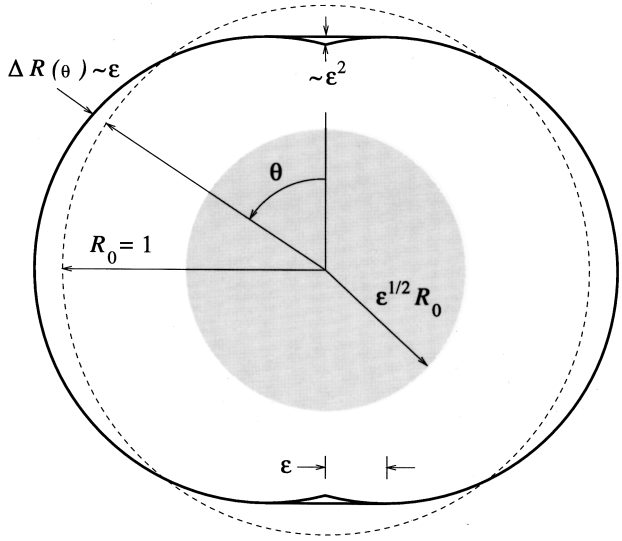

FIG. 3. The nearly circular Bunimovich stadium billiard $R(\theta)=R_0+\epsilon\Delta R(\theta)$ with $\epsilon=0.3$. The straight line segments have length 2ϵ. The area of the stadium is the area of the dashed circle with $R_0=1$. Straight-line orbits that cross the shaded region are affected by the $l=0$ resonance. Several relevant lengths are shown.

$$L_0(\theta,\theta')=2\left|\sin\frac{\theta-\theta'}{2}\right| \quad (9)$$

and

$$L_2(\theta,\theta')=\left|\sin\frac{\theta-\theta'}{2}\right|[\Delta R(\theta)+\Delta R(\theta')]. \quad (10)$$

C. Pedagogical case: Period-two resonance

We start with the lowest resonance, which is usually period one or two. These are typically the most prominent resonances. Of course, the definition of what a period is depends on the choice of surface of section. In standard action-angle coordinates, period one means that $\Theta_{11}=2\pi\equiv 0$, that is, the leading order $S\Phi$ point is $\theta'=\theta$. Similarly period two means that the $S\Phi$ point is $\theta'=\theta+\pi\equiv\theta-\pi$. These periods turn out to be special because S depends on two coordinates.

In addition we assume that S_0 is stationary at this period, i.e., that $S_0'(0)=0$, or $S_0'(\pi)=0$. Then, we may take $\psi_0=1$, that is, we are expanding about action $I_1=0$, so the states are entirely made up of ''low'' angular momenta. This means that ψ is slowly varying compared with the variation of the T operator.

In the example of the circle billiard, there is no period-one orbit. The simplest periodic orbits passes through the center of the circle and there are two bounces per period. From a classical perspective, after a weak perturbation the relevant orbits pass fairly close to the center of the circle. Straight-line orbits of this class, for the billiard of Fig. 3, pass through a shaded region at the billiard center. To carry out a pedagogical example which avoids the necessity of defining too many system-dependent constants, we specialize to the perturbed circle.

1. Case: $k\epsilon \ll 1$

Assume for the moment that $k\epsilon$ is small. In the billiard example this means that the wavelength is long compared to the shift of the boundary $\epsilon \Delta R$. Contrary to intuition, as we mentioned earlier this condition does not suffice to make perturbation results trivial. For the latter, it is necessary that $k\sqrt{\epsilon} \ll 1$.

With this condition, we may expand Eq. (4) as

$$T(\theta,\theta') \simeq -\sqrt{\frac{k}{2\pi i}(1+\cdots)}\exp\left(2ik - \frac{ik}{4}\delta\theta^2 + \cdots\right) \times [1 + ik\epsilon V(\theta) + \cdots], \tag{11}$$

where we may take $\delta\theta = \theta' - \theta - \pi$ as small, i.e., $\theta' \simeq \theta + \pi$. This follows since the stationary phase point is $\delta\theta = 0$. (We arrange things, to simplify the notation, so that $\theta' \simeq \theta + \pi$ rather than $\theta' = \theta - \pi$.) Here $V(\theta) = L_2(\theta, \theta + \pi) = L_2(\theta + \pi, \theta) = \Delta R(\theta) + \Delta R(\theta + \pi)$. The expansion of the second line of Eq. (11) relies on $k\epsilon$ small.

We also expand the wave function

$$\psi(\theta') = \psi(\theta+\pi) + \delta\theta\psi'(\theta+\pi) + \frac{1}{2}\delta\theta^2\psi''(\theta+\pi) + \cdots \tag{12}$$

and the phase

$$e^{i\omega(k)} = e^{i[2k + (1/2)\pi + \omega_0]}(1 + ik\epsilon E_m + \cdots), \tag{13}$$

where E_m is a shift of eigenphase to be determined and ω_0 is defined below. The $\frac{1}{2}\pi$ comes from the prefactor and Gaussian integral over $\delta\theta$. Using expressions (11), (12), and (13) in Eq. (7), and doing the $\delta\theta$ integral, we find the conditions for a solution.

First, we must require

$$\psi(\theta+\pi) = e^{i\omega_0}\psi(\theta). \tag{14}$$

In this case, ω_0 is 0 or π, since ψ must be 2π periodic. Insisting that the leading small terms vanish, we find that ψ must also satisfy the equation

$$-\frac{1}{k}\psi'' + k\epsilon V(\theta)\psi = k\epsilon E_m \psi. \tag{15}$$

This is a familiar equation, similar to the equation of a quantum pendulum, for motion in a periodic potential of strength $k^2\epsilon = (k\sqrt{\epsilon})^2$, and unit Planck. Alternatively, we may take a potential whose scale is unity and think of Planck's constant as $\hbar = 1/(k\sqrt{\epsilon})$. Thus $b = \sqrt{\epsilon}$ naturally appears.

Equation (15) is such a well known and thoroughly analyzed equation that the problem can be considered solved. Analytic methods are available if kb is small or large. For kb small, we can make an ordinary quantum perturbation expansion about the starting state of zero angular momentum $\psi = \text{const}$.

If kb is large, standard WKB theory makes the ansatz $\psi(\theta) = \exp[ikbf(\theta) + ig(\theta) + O(1/kb)]$, where $df/d\theta = f' \sim 1$ and $g \sim 1$. Using this approach gives

$$f'(\theta) = \pm\sqrt{E_m - V(\theta)}, \tag{16}$$

which is π periodic. If $E_m > V$ for all θ, we may quantize E_m by the condition

$$kb[f(\theta+\pi) - f(\theta)] = \pm kb\int_0^\pi d\theta\sqrt{E_m - V(\theta)} = \omega_0 + 2\pi m. \tag{17}$$

We call this the ''rotational'' case by analogy with the pendulum.

The usual next-order analysis gives $g(\theta) = i\frac{1}{2}\ln f'(\theta)$, which is customarily written as a prefactor $[E_m - V(\theta)]^{-1/4}$.

If $E_m < \max V(\theta)$, there will be potential wells, at least two in this case, with corresponding librational motion. This may be treated at various levels of approximation. If tunneling between the wells is neglected, there will be a quantization condition

$$kb\int_{\theta_{m-}}^{\theta_{m+}} d\theta\sqrt{E_m - V(\theta)} = \pi(m+\nu). \tag{18}$$

Here, the limits are the angles where the square root vanishes, the classical turning points, where the leading WKB approximation breaks down.

The Maslov index ν, usually $\frac{1}{2}$, is included. It can be found, for example, by the usual device of approximating V by a linear function in the neighborhood of the turning points, finding the Airy function solution, and using it to interpolate between the WKB solutions away from the turning point.

The wave function, sufficiently inside the turning points, is approximately given by

$$\psi = (E_m - V)^{-1/4}\sin\left(kb\int_{\theta_{m-}}^{\theta} d\tilde{\theta}\sqrt{E_m - V(\tilde{\theta})} + \frac{\pi}{4}\right). \tag{19}$$

This gives a twofold degeneracy, since there are librational levels in two identical wells. If desired, the exponentially small splitting of these levels can be estimated.

The condition $E_m = \max V(\theta)$ gives the separatrix of the motion between librational and rotational motion. Again, the simplest WKB approximation in the neighborhood of the separatrix can be corrected by well known if tedious methods.

Is it also straightforward to take into account symmetries which may exist, for example under $\theta \leftrightarrow -\theta$, or time reversal, which guarantees that eigenfunctions of Eq. (5) can be taken real.

2. Case: $k\epsilon \geqslant 1$

We now relax the condition $k\epsilon \ll 1$. Instead, we assume $k\epsilon \geqslant 1$, while $k\epsilon^{3/2} \ll 1$. Now it is not possible to expand the exponential representing the perturbation in the T operator. We can show, however, that the WKB solution of the preceding section remains valid.

Expanding the phase of the T operator about the point $\theta' = \theta + \pi$, as before, we obtain

$$S(\theta,\theta')=kL\approx 2k-\frac{1}{4}k\delta\theta^2+k\epsilon[\Delta R(\theta)+\Delta R(\theta')]. \tag{20}$$

[In Eq. (20) we replaced $|\sin\frac{1}{2}(\theta-\theta')|$ by 1, its stationary value, in $L_2(\theta,\theta')$.] The WKB ansatz is $\psi=\exp ikbf(\theta)$. The same prefactor will also be found, but we ignore it for now.

Returning to Eq. (7), and using this ansatz, we expand all functions of θ' about $\theta+\pi$. This is, $f(\theta')\approx f(\theta+\pi)+\delta\theta f'(\theta+\pi)$ to order $\delta\theta$ (since $kb\ll k$) and $\Delta R(\theta')\approx\Delta R(\theta+\pi)$ (since $k\epsilon\ll kb$). Doing the integral reduces Eq. (7) to

$$\exp[ikbf(\theta)+i\omega(k)] = i\,\exp\{i[2k+(kbf')^2/k+k\epsilon V(\theta)+kbf(\theta+\pi)]\}, \tag{21}$$

where $V(\theta)=\Delta R(\theta)+\Delta R(\theta+\pi)$ as before.

For Eq. (21) to hold, the phases of order kb must combine to give a constant ω_0, i.e., $f(\theta+\pi)=f(\theta)+\omega_0/kb$. At order kb^2 solution is possible provided $(f')^2+V(\theta)$ is a constant, which again we call E_m. Thus again

$$f(\theta)=\pm\int^{\theta}d\theta'\sqrt{E_m-V(\theta')}. \tag{22}$$

The lower limit can be chosen at our convenience. Notice $V(\theta)=V(\theta+\pi)\Rightarrow f(\theta+\pi)=f(\theta)+\text{const}$.

Everything goes through exactly as in the preceding section for the rotational states where $E_m>\max V$. For $E_m\leqslant\max V$, we have librational (or near separatrix) states where leading order WKB fails in the usual way.

However, we may proceed by almost the standard technique. Namely, for angles near the turning point, $k\epsilon(E_m-V)$ is small, even if $k\epsilon$ is not. *In this region,* therefore, we can expand as in Eqs. (11) and (12). The solution in this region interpolates between the regions where WKB is good. With some additional arguments, we obtain the usual WKB results for a potential V, including prefactors and Maslov indices.

In Fig. 4 we compare the results of this approximation with exact numerical determination of the wave functions. The exact results show the tunneling tails and Airy-function-like interpolation at the turning points, which we have not bothered to calculate in WKB theory.

We remark that in this case, the results are independent of the size of $k\epsilon$, that is, there are no corrections to the result of order $k\epsilon$, but only of order $k\epsilon^{3/2}$. Thus we show two wave functions that cannot be distinguished, with the same value of kb, but quite different values of kb^2, one smaller than unity, the other larger.

We remark that the $\delta\theta$ integral is effectively over a width of order $1/\sqrt{k}$. However, the center of the effective window of integration is not at $\delta\theta=0$, if we take into account the existence of f, but rather at $\delta\theta\simeq 2\sqrt{\epsilon}f'$. This shift will be small compared with the width if $\sqrt{\epsilon}\ll\sqrt{1/k}$, or $k\epsilon\ll 1$. Thus the term in ψ' of Eq. (12) does not contribute for $k\epsilon\ll 1$. If $k\epsilon\geqslant 1$, that condition does not hold, but in the turning point region where we are expanding, the shift is small because f' is small.

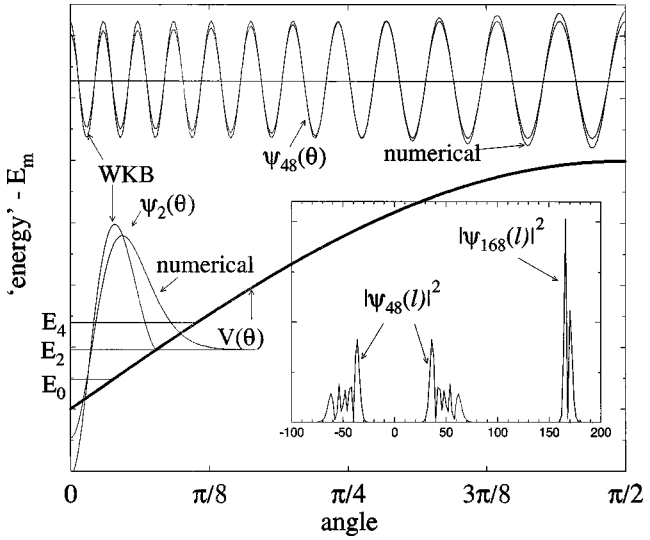

FIG. 4. Some results for low angular momentum states in the ''stadium'' billiard. The effective potential, $V(\theta)=|\sin\theta|$, vs angle. States and potential are symmetric about zero angle. Librational and rotational states are shown in our ''WKB'' approximation as well as the numerically exact case. The zero axis of the state is at WKB ''energy'' parameter E_m. $k\sqrt{\epsilon}=42.3$ is fixed. Inset: angular momentum representation of continuum states centered at angular $m=48$ and $m=168$.

Thus, we conclude that, even if $k\epsilon\geqslant 1$, so that $k\sqrt{\epsilon}\gg 1$, the usual WKB solution of Eq. (15) also solves Eq. (7).

D. Classical interpretation

We now give some classical implications of these results.

The action $S(\theta,\theta')$, the phase of the T operator, generates the surface of section map $\mathcal{T}(l',\theta')\rightarrow(l,\theta)$ by the equations $l=\partial S/\partial\theta$, $l'=-\partial S/\partial\theta'$. For the distorted circle billiard, this is explicitly $l=-k(\theta-\theta'+\pi)/2+k\epsilon\Delta R'(\theta)$; $l'=-k(\theta-\theta'+\pi)/2-k\epsilon\Delta R'(\theta')$. It is customary to desymmetrize this map a bit by setting $l=\tilde{l}+k\Delta R'(\theta)$. Then the map is

$$\tilde{l}=\tilde{l}'+k\epsilon V'(\theta'), \qquad \theta=\theta'-\pi-2\tilde{l}/k. \tag{23}$$

This map has been obtained by other methods [16,17].

If qualitative understanding is the main motivation, S may be simplified while keeping important physics intact. For example, we may regard Eq. (20) as exactly defining a map which we wish to study classically and/or quantally. A natural simplification is to use $\delta\theta=\theta'-\theta$. It seems natural to take $\Delta R=\cos\theta$ as the simplest possible periodic perturbation, which yields the well known Chirikov-Taylor or standard map [19,21]. Although this choice of ΔR does capture many phenomena of interest, we find that it is not typical in certain senses. We return to this point below.

Let the invariant loop be given by a formula $l_{\text{inv}}(\theta)$. This function will have to be two-valued if it is defined for a finite range of angle, and it can be single-valued if it is defined over the full range of angle. It must satisfy

$$\mathcal{T}(l_{\rm inv}(\theta'),\theta')\rightarrow(l_{\rm inv}(\theta),\theta).$$

In terms of the generating function, it must satisfy $l_{\rm inv}(\theta)=\partial S(\theta,\theta')/\partial\theta$ *and* $l_{\rm inv}(\theta')=-\partial S(\theta,\theta')/\partial\theta'$. Integrating this relation, and calling $F(\theta)=\int l_{\rm inv}(\tilde{\theta})d\tilde{\theta}$, we see that

$$F(\theta)=S(\theta,\theta^*)+F(\theta^*)+\text{const}, \tag{24}$$

where $\theta^*(\theta)$ is the angle such that $\partial S(\theta,\theta^*)/\partial\theta^*+F'(\theta^*)=0$.

If $S(\theta,\theta')+F(\theta')$ appears as a phase in an integral over θ', the stationary phase point is of course $\theta'=\theta^*$. Thus we see that we found an approximate invariant loop, $l_{\rm inv}(\theta)\approx k\sqrt{\epsilon}f'(\theta)=\pm k\sqrt{\epsilon}\sqrt{E_m-V(\theta)}$. More generally, this is the first term in a series expansion for $l_{\rm inv}$. Therefore, E_m is an approximate constant of the classical motion, which, like the energy, will be quantized.

The loop $l_{\rm inv}$ we have found so far is not truly invariant because it is approximate. Rather, it will be mapped into a new loop $\mathcal{T}(l_{\rm inv})\rightarrow l_1(\theta)\neq l_{\rm inv}(\theta)$. The area enclosed by the two loops is the same, since $\mathcal{T}$ is area preserving. It is possible to estimate the phase-space area of points inside l_1 but outside $l_{\rm inv}$. In the worst case, the area occupied by such ''turnstile'' points is proportional to $\epsilon^{3/2}$, which is a factor ϵ smaller than the area of $l_{\rm inv}$ itself. If the turnstile area is much smaller than $\hbar$, i.e., if $kb^3\ll 1$, it is reasonable to believe that $l_{\rm inv}$ is a good approximation for the purposes of quantum mechanics. This is true *even if no true invariant loop exists.* Of course, if a true invariant loop exists, it may be possible to make a correction to $l_{\rm inv}$, i.e., to find a higher-order approximation to the invariant loop such that the turnstile area is even smaller. This would then allow us to take $kb^3\geqslant 1$ and still obtain good results.

We thus see that the libration states with $E_m<\max V$, are to be identified with the resonance islands about the stable fixed points. The rotational states with $E_m>\max V$ are distortions of the unperturbed invariant loops $l_{\rm inv}=\text{const}$. If $E_m\gg\max V$, this distortion is small. However, this approximation breaks down at some stage because there will be higher resonances which must be taken into account.

E. Energy and quasienergy quantization

We have seen that there is a WKB quantization of the approximate classical constant of the motion E_m. This leads to an expression for the quasienergy ω, in the circle case, of Eq. (7),

$$\omega=\omega(k)=2k+k\epsilon E_m+\omega_0+\pi/2. \tag{25}$$

Table I gives a comparison of numerical quasienergies with those given by this formula for a number of states. The numerical method used is that introduced in Ref. [22] to solve the standard map. Starting with an approximate wave function, a long time series is obtained by applying the T operator repeatedly. Fourier transform of this series yields the eigenphases and eigenfunctions. The T operator can be applied very efficiently if it can be factored into a part dependent only on $\theta-\theta'$, and a part whose phase is a sum of a function of θ and a function of θ'. The latter is not strictly true for the nearly circular billiard, but it is true to order b^3. Note also that we are comparing our method to the numerical solution of Eq. (5), and not to the numerical solution of the Helmholtz equation. The question of how well the T operator corresponds to the billiard has already been addressed [8].

TABLE I. Numerical quasienergies and quasienergies calculated from the perturbation theory ($\omega_{\rm PT}$), compared for states with different m, but all belonging to the same period-two resonance. ΔR corresponds to the stadium billiard. WKB ''energy'' parameter is also given, $k=1000$ and $\epsilon=6.1\times10^{-4}$ are fixed.

m	E_m+4/π	ω numerical	$\omega_{\rm PT}$
1	0.2083	5.6199	5.6310
3	0.4321	5.7701	5.7680
5	0.6053	5.8726	5.8741
7	0.7544	5.9661	5.9654
9	0.8878	6.0465	6.0471
11	1.0097	6.1222	6.1218
13	1.1223	6.1904	6.1908
15	1.1272	6.2553	6.2550
17	1.3254	0.0317	0.0319
19	1.4175	0.0885	0.0884
21	1.5040	0.1411	0.1413
23	1.5852	0.1912	0.1911
25	1.6614	0.2376	0.2378
27	1.7327	0.2814	0.2814
29	1.7989	0.3217	0.3220
31	1.8599	0.3592	0.3593
33	1.9152	0.3924	0.3932
35	1.9638	0.4243	0.4229
37	2.0010	0.4461	0.4458

The results are quite good, that is, the errors are small compared with the separation of the levels. The separation is of order

$$k\epsilon(E_{m+1}-E_m)\simeq\sqrt{\epsilon}. \tag{26}$$

The *energy* levels are given by choosing values of k which solve

$$\omega(k)=2\pi n. \tag{27}$$

This has solutions $k=k_{n,m}$ as for the perfect circle (see Table II). [For $\Delta R=0$, $\omega=2\pi n$ reduces to $2k+m^2/k+\pi m=(n-\frac{1}{4})2\pi$, where m is the angular momentum. This is equivalent to Debye's approximation to Bessel's function, valid for k large and m/k small.]

For fixed m, the variation of ω with k from the first term, $2k$, of Eq. (25) dominates. Thus we have

$$k_{n+1,m}\simeq k_{n,m}+\pi. \tag{28}$$

The error in the determination of a given energy level is not, however, small in comparison with the mean spacing in energy of *all* the levels. The levels so far found are a small fraction of the levels in a given energy range. There are levels belonging to larger angular momenta and smaller radial wave number in the same range. In terms of k, the level spacing of all the levels is about $2/k$, which is the order of the size of the errors committed and is the order of the absolute error of a given level.

TABLE II. Energies k with different quantum numbers n, but the same $m=11$, computed numerically and found solving Eqs. (25) and (27) (k_{PT}). ϵ as in Table I.

n	E_m+4/π	k numerical	k_{PT}
319	1.0108	998.3207	998.3213
318	1.0128	995.1784	995.1789
317	1.0148	992.0358	992.0364
316	1.0169	988.8934	988.8940
315	1.0190	985.7509	985.7515
314	1.0210	982.6085	982.6090
313	1.0231	979.4660	979.4666
312	1.0252	976.3235	976.3241
311	1.0273	973.1810	973.1816
310	1.0294	970.0387	970.0392

It would be nice to have a theory giving the energy levels with absolute accuracy less than the mean level spacing, of course. Primak and Smilansky [23] have discussed how this might be possible within the framework of quasiclassics, even though the errors made in arriving at the T operator and doing stationary phase integrals are of order $\hbar$. This, however, involves finding the mean density of levels to better than leading-order accuracy by a separate calculation, and combining it in a particular way with sums over periodic orbits.

Nevertheless, the results we have obtained are very useful and contain nearly all that is desired. The energy levels are classified into groups. Matrix elements of smooth operators are large only between levels in the same group. The error is small compared with the spacing of the levels in the same group. The overall statistics of the levels on the scale of the mean level spacing as well as long-range correlations of energy levels are also given correctly [24].

F. Wave functions and localization

The ''wave functions'' are also given to good approximation, as we showed above. Of course, there can be an accidental degeneracy between energy levels coming from small angular momenta, such as we have discussed above, and energy levels carrying large angular momenta, which we will calculate below. Then, the true eigenstates will be some appreciable admixture of large and small angular momenta. Lazutkin's rigorous results also allow for this possibility [5]. However, the matrix elements of the T operator between widely different angular momentum states are exponentially small. This means that such accidental degeneracies will be rare.

In the presence of symmetries, there will be degeneracies which are exact in the absence of the exponentially small coupling. An example is the states associated with the periodic wells discussed above. In the presence of time reversal invariance, the states associated with positive angular momenta are degenerate with those of the corresponding negative angular momenta, again with very small splitting due to tunneling between the two momentum regions. If desired, this splitting can be estimated within the framework of our theory. This is not entirely trivial as it requires a study [24] of ''resonance assisted tunneling'' which is an analog of ''chaos assisted tunneling'' [25].

The wave function so far found contains only small angular momentum components. This can be seen explicitly in angular momentum representation, determined schematically from the integral

$$\psi_l=\int d\theta[E_m-V(\theta)]^{-1/4}\exp(ik\sqrt{\epsilon}f(\theta)-il\theta). \quad (29)$$

If $k\sqrt{\epsilon}\gg 1$, $S\Phi$ may be employed. Then, it is found that only the neighborhood of the angles solving

$$k\sqrt{\epsilon}f'(\theta)=k\sqrt{\epsilon}\sqrt{E_m-V(\theta)}=l \quad (30)$$

contribute appreciably. A range of l centered roughly at $k\sqrt{\epsilon E_m}$ and of width of order $k\sqrt{\epsilon\ \max V}$ can satisfy this condition, so $k\sqrt{\epsilon\ \max V}$ is a sort of localization length for this kind of wave function. Note that only librational states of the lowest resonance have much overlap with zero angular momentum.

Outside this range, the angular momentum components of the wave function decay exponentially for smooth V. For the stadium case, $V=|\sin\theta|$, it may easily been seen that the decay is as l^{-4}, rather than exponential. This makes it necessary to use nonstandard definitions of the localization length [16,19,20] if statistical results are to be calculated.

Carrying out the integral of Eq. (29) gives an expression

$$\psi_l=\sum_{\theta_a}|V'(\theta_a)|^{-1/2}\exp\ i\left(k\sqrt{\epsilon}f(\theta_a)-l\theta_a+\nu_a\frac{\pi}{2}\right), \quad (31)$$

where θ_a are the solutions of Eq. (30).

Notice that, if as expected, V' has different signs at the different θ_a, there will be an additional Maslov phase index ν_a distinguishing the two solutions. Also notice that the singularity of the prefactor at the classical turning point in angle, where $V(\theta)=E_m$, has disappeared. Instead, the angular momentum regions which make V' vanish, which are the turning points in the angular momentum representation, are singular, and should be treated by a technique going beyond the first WKB approximation. This shift of the region of breakdown of the leading WKB approximation under Fourier transform is the basis of Maslov's treatment [26] of this subject.

The solutions thus found are *localized* in angular momentum space. From the point of view of KAM theory and perturbation theory, this is unremarkable. First, the KAM theory predicts that the classical motion is also localized as in Eq. (30). Second, matrix elements of the T operator in the angular momentum representation are small away from the diagonal. In fact, consider $T_{ll'}=(2\pi)^{-1}\int\int d\theta d\theta' e^{-il\theta}T(\theta,\theta')e^{il'\theta'}$, where $(l-l')/k$ is of order unity. The θ' integral will be stationary near $dS_0(\theta-\theta')/d\theta=\hbar l'$, and the θ integral will be stationary near $dS_0(\theta-\theta')/d\theta=\hbar l$. Thus, there is no stationary phase point for this rapidly oscillating double integral, and we can conclude that the integral is very small.

Such a band diagonal matrix $T_{ll'}$ is studied in localization theory [22] and is effectively the subject of the papers [16,17,19,20] cited earlier. Since the classical long-time be-

havior is diffusive, in the non-KAM cases, the mechanism for localization was not obvious. If stationary phase arguments are sufficient to get the width of the band, as we have assumed, the problem is solved. The cited papers are concerned with the non-KAM case that $k\epsilon^{M/2}$ is sufficiently large that stationary phase breaks down.

It should be emphasized that our whole theory is based on the assumption, validated self-consistently, that the eigenstates are localized in angular momentum.

G. Adiabatic approximation

The full two-dimensional wave function can also be found in WKB approximation. This is quite interesting and will be the subject of a further communication. The basic result is that the full wave function $\Psi(r,\theta)$ is a generalization of the wave function used in the Born-Oppenheimer theory. In other words, for the case considered in the present section, it is something like $\Psi(r,\theta)=\Phi(r;\theta)\psi(\theta)$, where the ''fast'' variable is r and the ''slow'' variable is θ. To leading approximation, the θ dependence of Φ is parametric: derivatives with respect to the θ dependence of Φ are negligible.

It is not too surprising that the present method is related to the adiabatic approximation. Our method of stationary phase systematically orders the phases appearing in the problem by how rapidly they vary. The Born-Oppenheimer approximation distinguishes between rapidly varying and slowly varying coordinates. In some cases our method is exactly equivalent to Born-Oppenheimer. In other cases, our method produces adiabatic invariants [10]. These have in the past been used to quantize certain states [15,27]. In other cases, we obtain a generalization of the standard adiabatic approximations.

IV. HIGHER RESONANCES

We turn to a discussion of the states where $\psi_0=\exp il\theta$, with $|l|\gg 1$. As we have just remarked, such a state cannot communicate directly via the T operator with its negative l counterpart. It is true that time reversal symmetric systems will be symmetric under $l\to -l$, so that true eigenstates will be an even or odd mixture of these states. However, the even and odd eigenstates will be practically degenerate, and we can consider one sign of angular momentum only to get the main effect.

Suppose we wish to find eigenstates centered near angular momentum l. One way is to reduce the problem to the one just solved by making a sort of gauge transformation, in which the eigenstates $\psi\to e^{il\theta}\tilde{\psi}$, $T\to e^{-il(\theta-\theta')}\tilde{T}$, and $\tilde{T}\tilde{\psi}=\tilde{\psi}$. We may then assume that $\tilde{\psi}$ is slowly varying.

There are some complications in this case as compared with the case where $l=0$. First, to leading order, the stationary phase point is given by $dS_0(\theta-\theta')/d\theta=l/k$, namely at $\theta-\theta'=-\Theta_l$, say. However, Θ_l has no particular relationship to 2π, that is, it will be an irrational multiple of 2π, unless l is specially chosen. (The negative sign in the definition of Θ_l is purely for orthographic convenience.) If l is an integer, this cannot happen except at special symmetry values such as $l=0$.

There are two possibilities. Either Θ_l is very close to a rational multiple of 2π, $2\pi p/q=\Theta_{pq}$, where q is not too big, or it is sufficiently far from such a multiple. It is the job of the theory to tell us what ''very close,'' ''not too big,'' and ''sufficiently far'' mean in detail. The resonant case is the one where Θ_l is ''close'' to a rational, the KAM-set case has Θ_l ''far'' from a rational.

The first case we deal with is that of a resonance of period q, which involves destruction of the original invariant tori and their replacement by new tori with a different topology. These are represented on the surface of section by a period q island chain. The second, nonresonant, case deals with invariant tori which are somewhat distorted by the perturbation but which keep the same topology.

In the resonance case we consider states whose angular momentum components are not too far from l_{pq} where the leading stationary phase point corresponding to $l=l_{pq}$ is Θ_{pq}. Note that $\tilde{\psi}$ will not be 2π-periodic since l_{pq} is not generally integer.

The ansatz made before,

$$\tilde{\psi}=\exp(ikb\tilde{f}(\theta)), \tag{32}$$

does not lead to a solution. Indeed, expanding $\tilde{T},\tilde{f}$ about Θ_{pq}, we do the θ' integral and find

$$\exp\{ik\epsilon[c^2\tilde{f}'(\theta+\Theta_{pq})^2+L_2(\theta,\theta+\Theta_{pq})]\}$$
$$=e^{-i\alpha}e^{i\omega}\exp\{ikb[\tilde{f}(\theta)-\tilde{f}(\theta+\Theta_{pq})]\}. \tag{33}$$

Here c is determined from the second derivative of S_0. The phase α is obtained by doing the θ' integral. To satisfy $\tilde{T}\tilde{\psi}=\tilde{\psi}$, we must require

$$\tilde{f}(\theta+\Theta_{pq})=\tilde{f}(\theta)+\text{const} \tag{34}$$

and

$$c^2(\tilde{f}')^2+L_2(\theta,\theta+\Theta_{pq})=\text{const}. \tag{35}$$

These two conditions cannot in general be simultaneously fulfilled. Equation (34) implies that $\tilde{f}'$ is q-periodic, i.e., periodic with period Θ_{pq}, but only a special choice of perturbation L_2 in Eq. (35) will be q-periodic, if $q\neq 2$.

We therefore improve the ansatz and take

$$\tilde{\psi}=\exp\{ikd[bf_1(\theta)+b^2f_2(\theta)+\cdots]\}. \tag{36}$$

Introducing $d=-2S_0''(\Theta_{pq})$ simplifies later formulas. Treating b^2f_2 as slowly varying, in this order of the calculation, we find the conditions

$$f_1(\theta+\Theta_{pq})=f_1(\theta)+\text{const} \tag{37}$$

and

$$f_2(\theta)-f_2(\theta+\Theta_{pq})=f_1'^2+W_{pq}(\theta)-E_m, \tag{38}$$

where E_m is a constant to be determined and $W_{pq}(\theta)=L_2(\theta,\theta+\Theta_{pq})/d$. The constant E_m leads to a phase $kd\epsilon E_m$.

Define the q average of a function of angle by $\overline{F}_q(\theta)=1/q\sum_{r=1}^{q}F(\theta+r\Theta_{pq})$. The Fourier components of $\overline{F}_q$ are

just those of F with all components whose index is not divisible by q discarded. Making a q average of Eq. (38) leads to

$$f_1' = \pm\sqrt{E_m - \overline{V}_q(\theta)}, \tag{39}$$

where $\overline{V}_q$ is the q average of W_{pq}.

The function f_2 cannot be completely determined at this level, since an arbitrary q-periodic function can be added to it and not affect Eq. (38). Calling $V_{pq} = W_{pq} - \overline{V}_q$, we can solve for the remaining part of f_2. This function has no Fourier components whose index is divisible by q. The solution of $f_2(\theta) - f_2(\theta + \Theta_{pq}) = V_{pq}$ can be made by Fourier transform of the equation. An explicit solution can also be written

$$f_2(\theta) = \frac{-1}{q}\sum_{r=0}^{q-1} r V_{pq}(\theta + r\Theta_{pq}). \tag{40}$$

This suggests making the following transformation. Let $\tilde{\psi} = e^{ik\epsilon d f_2}\hat{\psi}$ and consider the corresponding operator $\hat{T}$ which may be approximated by

$$\hat{T}(\theta,\theta') \simeq \sqrt{\frac{kd}{2\pi i}}\exp\left(2ikd - \frac{ikd}{4}\delta\theta^2\right) \times \exp\{ikd\epsilon[W_{pq}(\theta) + f_2(\theta + \Theta_{pq}) - f_2(\theta)]\}. \tag{41}$$

The second exponential is $\exp(ikd\epsilon\overline{V}_q)$. If $kd\epsilon$ is small, we expand $\hat{\psi}$ as in Eq. (12) above. Thus, $\hat{\psi}$ solves the periodic potential problem with potential $\overline{V}_q$ and effective Planck's constant $1/kd\sqrt{\epsilon}$.

If $kd\epsilon$ is of order unity or larger, the argument goes through as before, so that we have obtained the solution to the resonant case. This also gives the prefactors, as before.

We show in Figs. 5(a)–5(c) some invariant loops for higher resonances $pq = 1{,}4; 7{,}28$. The numerical loops obtained by iterating the SS map are compared with the theoretical $l_{\rm inv} = l_{pq} + bf_1' + b^2 f_2'$.

There are several things of note. First, the contribution of f_2 becomes relatively more important for higher q. This is for two reasons. Most obviously, $(\epsilon\overline{V}_q)^{1/2}$ systematically becomes smaller with increasing q, and eventually becomes smaller than ϵ, so that the small resonance islands 'float' on a relatively large ''wave'' supplied by f_2. Second, f_2 can itself become rather large. This can be seen from formula (40), which can become fairly big if the terms in the sum add ''in phase.'' This effect is most pronounced near a large resonance island, e.g., at $q = 2$. Then the term $\epsilon f_2'$ for p/q slightly greater than $\frac{1}{2}$ with q large must be almost as big as $\sqrt{\epsilon} f_1'$ for $q = 2$.

Another remark is that higher-order terms more generally are relatively more important for large q. This can be seen in the results. These deviations are apparent in this case of rather large $b = 0.26$, $b^2 = 0.07$, $b^3 = 0.02$. Again this is because $\overline{V}_q$ is becoming smaller, while the higher-order corrections presumably remain roughly constant. The observed deviations are apparently of order b^3. Of course, if kb^3 is small, the quantum results will not be affected by the deviations.

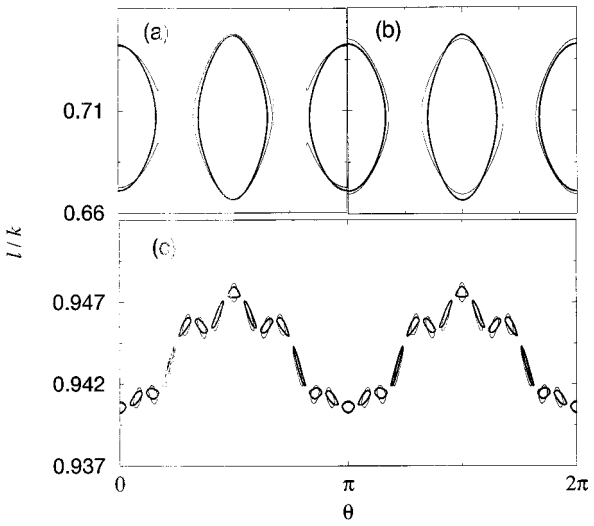

FIG. 5. Approximate and exact invariant loops at the fourth-order and twenty-eighth-order resonances. The value of ϵ is 0.05 in (a),(b) and 0.01 in (c). η is 0.7 in (a),(b) and 0.19 in (c). (a) includes the corrections to order b^3, while (b) and (c) are correct only to order b^2. Thus we have verified that the main discrepancy between (b) and the exact loop is accounted for by the next-order correction. Higher period resonances have relatively, although not absolutely, bigger corrections from higher orders in $\sqrt{\epsilon}$.

The case $pq = 3{,}28$ shows an impressive set of wriggles coming from f_2', which are quite well predicted theoretically. We have not tried to find the ''best'' value E_m corresponding to a given classical numerical $l_{\rm inv}$, except by trial and error. This may contribute somewhat to the discrepancies. The islands are of different shape, however, and vary in height from one island to the next. These effects cannot be reproduced in the order kb^2 but are order kb^3 effects.

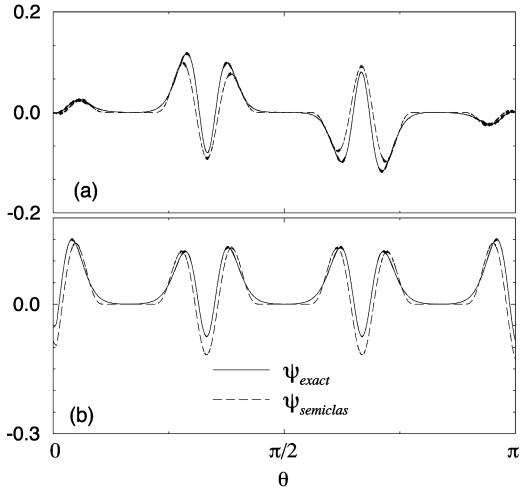

FIG. 6. The exact and approximate wave functions for the 1,3 resonance, with $k = 4032$, $\epsilon = 6.79 \times 10^{-4}$. The theoretical approximate wave function is $\psi = \cos(kdbf_1)\sin(l\theta + kdb^2 f_2)$. The rapidly varying dependence on $l\theta$ is removed by locally averaging $\psi \sin l\theta$ (a) and $\psi \cos l\theta$ (b).

In Fig. 6 we show a quantum state for the 1,3 resonance. What is plotted is $\tilde{\psi}$, that is, the rapidly varying factor $\exp il\theta$ is removed.

A. Quantization conditions

The quantization conditions are somewhat modified, since the central angular momentum l_{pq} is not usually integer. We require that $\exp[il_{pq}\theta+ikdbf_1(\theta)]$ be 2π-periodic. For the rotational levels, $E_m>\max \overline{V}_q$, we have a modified quantization condition

$$\int_0^{2\pi} d\theta\sqrt{E_m-\overline{V}}=2\pi(m+\delta)/kbd, \tag{42}$$

where $-\delta$ is the fractional part of l_{pq}.

If there are solutions of

$$kbd\int_{\theta_{m-}}^{\theta_{m+}} d\theta\sqrt{E_m-\overline{V}_q(\theta)}=\pi(m+\nu), \tag{43}$$

where the integral is between turning points straddling a minimum of the potential, ν again being the Maslov index, there will be librational solutions repeated q times. There are then q nearly degenerate states, and the phase shift from one well to the next will be slightly different than Θ_{pq} in such a way as to make the total wave function 2π-periodic.

These quantization conditions make the area of the loops an integer number multiplied by Planck's constant plus the correction due to Maslov indices. Thus, it is equivalent to EBK quantization.

B. When the resonant solution is applicable and needed

The crucial scale I_{pq} of Sec. I D is usually provided by $\overline{V}_q$, which provides the estimate

$$I_{pq}=\max|\overline{V}_q|. \tag{44}$$

We assume without loss of generality that $\overline{V}_q$ has vanishing mean.

If librational states exist, then we clearly need the resonant solution. This requires $I_{pq}\geqslant q^2/k^2\epsilon$.

If $\overline{V}_q$ is very small, the smallest rotational E_m is of order $E_m\sim(kbd)^{-2}$. Thus, if $I_{pq}\geqslant(kbd)^{-2}$ we cannot neglect it. If I_{pq} is much less than this value, we can neglect $\overline{V}_q$ and the states in this angular momentum neighborhood are candidates to be treated by methods valid for the KAM set.

We remark on several cases which are distinguished by the way the Fourier components of $L_2(\theta,\theta+\Theta_{pq})$ behave.

For the example of a deformed circle, if ΔR is a pure low harmonic, or sufficiently close to one, e.g., $\Delta R=\cos\theta$ or $\Delta R=\cos 2\theta$, then $\overline{V}_q=0$ for $q>2$. Of course, it is immediately recognized that these cases are to first approximation a shifted circle and an integrable ellipse, respectively, so that they can be transformed to a perturbation about an integrable system with perturbation parameter ϵ^2 rather than ϵ.

From our perspective, if $\overline{V}_q$ vanishes or is sufficiently unusually small, one should determine if some higher power of ϵ gives resonant behavior. This is not hard to do in the simplest cases, but is tedious for more complicated cases. For example, the order q resonance of the standard map for small ϵ will have width parameter proportional to $\epsilon^{q/2}$, since the effective resonant coupling is $(\epsilon\cos\theta)^q$. Cases where the Fourier expansion of ΔR is a finite Fourier series will have a rather complicated condition which is difficult to treat in generality.

Analytic perturbations usually have Fourier coefficients which drop off exponentially with index. This leads to an estimate $I_{pq}\sim c^q$, with c less than 1. If $c>\epsilon$, we can expect that the treatment given above is adequate and the leading-order expression for the resonance is proportional to $\sqrt{\epsilon}$. If not, we either have to work out the higher-order effects or take k sufficiently small that we can ignore the resonance entirely.

Finally, there are the cases with nonanalytic S_2, whose Fourier coefficients drop off as a power, say. The billiard $\Delta R=|\sin\theta|$ has $\overline{V}_q\sim q^{-2}$, for example, where KAM theory breaks down completely. We can generally expect our leading-order solution to be valid for these cases. In this case, the entire phase space is best regarded as filled with resonances [28], leaving no KAM set.

C. Close to a large resonance

The following issue is to some extent still unresolved. Suppose $\overline{V}_q$ is sufficiently large that Eq. (42) holds nontrivially. Consider, for example, the resonance labeled 15,32 in Fig. 2. For values of $E_m>\max\overline{V}_q$, the predicted rotational invariant loops are well described by the low order resonance formula, although that formula does not give higher-order resonances. Assuming E_m large enough to expand in $\overline{V}_q$, the invariant loop is given by $bf_1'+b^2f_2\approx b\sqrt{E_m}(1-\frac{1}{2}\overline{V}_q/E_m)+b^2f_2$. This may be shown to approximate the nonresonant solution, also called f_2 but for a nonresonant value of l, found in the next section [24]. However, even if one is quite close to the separatrix, where one expects the resonant formula to be best, the resonant or nonresonant formula at the appropriate l value is not bad. Even the formula for a higher-order resonance which seemingly lies in a region strongly affected by the big resonance does qualitatively quite well, although there are some small discrepancies.

This seems to present a contradiction. Close to the big resonance, e.g., $p=1$, $q=2$, a secondary island chain from a high-order resonance follows well a loop given by $b\sqrt{E_m-\overline{V}_2}$ *and also a loop with seeming smaller variation* ϵf_2, corresponding to a resonance pq. However, according to Eq. (40), f_2 can become quite large for large q, which can compensate for the fact that $\epsilon\ll\sqrt{\epsilon}$.

V. NONRESONANT CASE, QUANTIZATION OF THE KAM SET

In this case, we start with a value of integer angular momentum l, such that Θ_l is not resonant. There is no term f_1 and the solution begins with f_2. The condition for f_2 is (for the deformed circle)

$$f_2(\theta+\Theta_l)-f_2(\theta)=\Delta R(\theta+\Theta_l)+\Delta R(\theta). \tag{45}$$

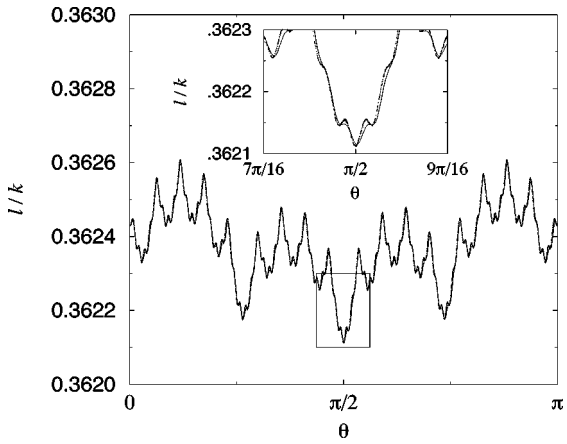

FIG. 7. The golden mean invariant torus of the smoothed billiard, approximated to order ϵ (solid line), $\cos\frac{1}{2}\Theta_l+\epsilon f_2'(\theta)$, and numerically exact classical map (dots, 20 000 iterations). Parameters are $\epsilon=10^{-4}, \eta=2\times10^{-2}$. The inset enlarges a cusplike portion of the figure to display the degree of failure of the approximation. This torus has $\Theta_l=2\pi w=\pi(\sqrt{5}-1)$, and it is ''farthest'' from a low-order rational. Such a torus is not really smooth, and is not describable by a convergent power series. It is ''transversely smooth,'' which is enough to control mathematically its properties [5]. One can pick out the closest resonances, e.g., 5/8, 8/13, etc., given by the Fibonacci series. Presumably, as η is decreased, this is the last torus remaining in the KAM set.

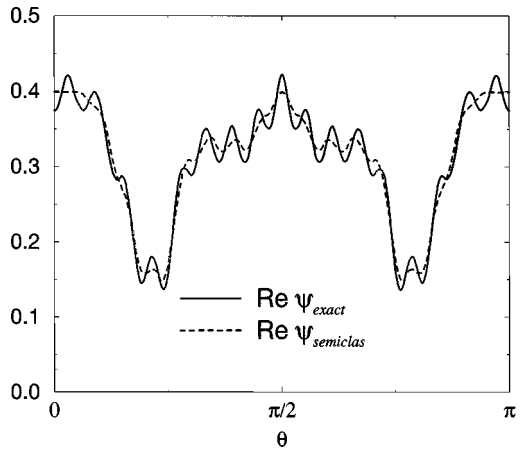

FIG. 8. Numerically exact and our approximate wave functions for the parameters of Fig. 7. The factor $e^{il\theta}$ has been removed and the real part of the wave function is shown.

This may be solved in terms of Fourier components of ΔR, denoted ΔR_r. The result is

$$f_2(\theta)=\frac{1}{2\pi}\sum_r \frac{e^{ir\Theta_l}+1}{e^{ir\Theta_l}-1}\Delta R_r e^{ir\theta}. \tag{46}$$

This sum will be well behaved, provided ΔR_r drops off sufficiently rapidly with r, and is small when the denominator is small. The denominator is small, when $r=q$ and Θ_l is close to a rational Θ_{pq}, with q not too large. The denominator then approximates $i(q\Theta_l-2\pi p)$. If this condition does not hold, the resonance treatment of the preceding sections must be employed.

At this order of the calculation, there is no shift in the energy levels for the KAM set states.

Formula (46) has been tested numerically. In Fig. 7 we plot $f_2'(\theta)$, the derivative of Eq. (46), and compare it with the numerical invariant loop, obtained by propagation of an orbit. It is apparent from this that the invariant loops are not totally smooth and featureless. KAM theory controls the singularity of these loops, although they cannot be represented by a convergent power series. In Fig. 8 we give the corresponding quantum state which captures most of the numerically exact results even in this leading order.

VI. HIGHER-ORDER CALCULATIONS

We briefly mention how one goes to higher order in the calculation. Some details are put in the Appendix. We confine attention to the resonant case.

Basically, one makes the ansatz

$$\psi=\exp\{i[l_{pq}\theta'+k(bf_1+b^2f_2+b^3f_3+\cdots)]\}. \tag{47}$$

Instead of expanding S_0 to second order in $\delta\theta=\theta'-\theta-\Theta_{pq}$, f_1 to first order, f_2 to zeroth order, and neglecting f_3, we expand, step by step, S_0 to the Mth order, f_1 to the $(M-1)$th order, $\ldots, f_M$ to the zeroth order, and neglect f_{M+1}, where $M\geqslant 2$, and $kb^{M+1}\ll 1$. The perturbation, which we may call $\epsilon S_2(\theta,\theta')+\epsilon^2 S_4(\theta,\theta')+\cdots$, is expanded according to the same rule. The order r calculation determines the nonperiodic part of f_r and the periodic part of f_{r-1}.

The resulting integral is not done exactly, but by stationary phase in the neighborhood of $\delta\theta=0$. The stationary point is regarded as shifted by $\Delta\Theta_{pq}$ which can be expressed as a power series in b. The b^r term in this shift is bigger than the width, $1/k$, of the effective region in the $\delta\theta$ integral if $kb^r\gg 1$. A sample calculation is done in the Appendix. Figure 5 shows a case where order kb^3 corrections are included. Evidently, turning-point corrections are important in this order, but we have not bothered finding them.

There are interesting questions about how this method breaks down if there are unusually large derivatives of S_2, but we shall not address them in this paper.

VII. OTHER PERTURBED INTEGRABLE SYSTEMS

There are an infinite number of integrable systems whose perturbations can be studied. Examples in the literature include coupled anharmonic [29] and Morse oscillators [31]. Our method must be modified to study coupled harmonic oscillators, usually studied by the Birkhoff-Gustavson method [6,33], since the winding number is constant in that case.

Rectangular billiards in a weak magnetic field, or with Aharonov-Bohm flux lines, have been of great recent interest [34]. These billiards can also be solved by our method, with new, unexpected, and rather striking results. Basically they show that the purely quantum phase can give localization even though the classical mechanics is totally unaffected. Although Anderson localization has long been known to be an effect of this kind, to our knowledge this is the first example where analytic formulas are given. This case will be published elsewhere.

Another interesting billiard is the weakly deformed rectangular billiard [35]. As an example of recent interest [36],

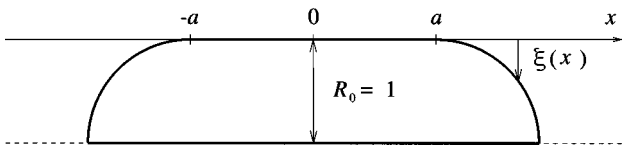

FIG. 9. Half of the stadium billiard which is compared to the straight channel, an integrable system.

consider the trapezoidal billiard, whose sides consist of the x axis, $0 \leqslant x < a$, and the vertical segments, $0 \leqslant y < b - \epsilon$ $(x = 0)$ and $0 \leqslant y < b + \epsilon$ $(x = a)$, and the slightly sloping segment connecting the upper ends of the vertical segments.

The results are much the same as before. We just mention what is found for $\epsilon \overline{V}_q$. This potential is periodic in x, with period $2a/q$. In one period, the potential is an isosceles triangle ''tent'' of zero mean and peak absolute value ϵ/q^2. Note that x is continued outside the physical region. This continuation is similar to using action-angle variables for the problem.

In fact, for any reflection symmetric perturbation ΔR of the circular billiard, there is a corresponding perturbation of the rectangular billiard, in which only one side of the billiard is perturbed.

VIII. NO SMALL PARAMETER

In this section we briefly consider some cases where a class of states of a system can be found, even though there is no small parameter. We will give the details elsewhere. In fact, the first application of the present method was the discussion of a class of states which turned up in the context of ray splitting [37]. A second important class of states are the whispering gallery modes, first discussed by Keller [27]. Our method easily reproduces these results. Keller's method relies on first finding an adiabatic invariant. This invariant drops out of our approach as a by-product.

Consider the well known example of the standard stadium billiard [38–40], which has a straight side of length $2a$, as in Fig. 9. The side a is arbitrary, but not too small. We want to study the ''bouncing ball'' states which have low linear momentum *parallel* to these sides, but a high *perpendicular* momentum. These states remain between the parallel sides and do not get out into the endcaps.

Reduce the billiard to its upper half by symmetry. Take as nominal integrable system the infinite channel of width $R_0 = 1$ as shown. We take as SS the upper boundary of the billiard, and use position x, measured from the symmetry point, on the upper channel to label position on the surface of section. The billiard boundary is described by distance ξ from the upper nominal channel, $\xi(x) = 0$, $|x| < a$, $\xi(x) \approx (|x| - a)^2/2$, $|x| > a$. The integrable action is, for states odd under reflection about the horizontal symmetry line,

$$\frac{S_0(x - x')}{\hbar} = 2k \sqrt{1 + \frac{1}{4}(x - x')^2} \approx 2k + \frac{k}{4}(x - x')^2. \tag{48}$$

States even about this line have the same action with an additional term $\pi\hbar$, that is, the T operator has an additional overall negative sign for the even states. The change of sign

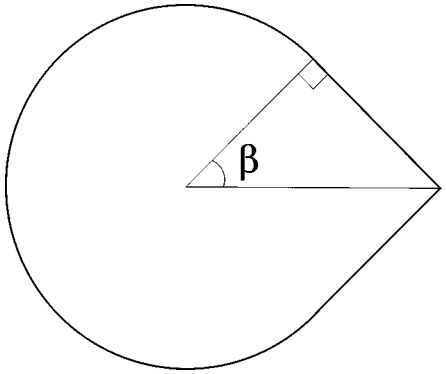

FIG. 10. The ice cream cone billiard with parameter $\beta = \pi/4$.

of the second derivative, as compared with the circle, means that the attractive and repulsive regions of the potential are reversed in comparison with the perturbed circle case. We can approximate S_2 by

$$S_2(x, x') = -k[\xi(x) + \xi(x')]. \tag{49}$$

Set

$$V(x) = -S_2(x, x)/k$$

$$= 0, \quad |x| < a \tag{50}$$

$$= (|x| - a)^2, \quad |x| > a. \tag{51}$$

We look for a solution

$$\psi = \exp(ikf(x)), \tag{52}$$

where f is slowly varying. One finds, upon approximating $f(x') = f(x) + f'(x)(x' - x)$,

$$f'(x) = \pm \sqrt{E_m - V(x)}. \tag{53}$$

Thus, we look for a solution of a particle in the potential well $V(x)$, and with effective Planck's constant $1/k$. If E_m is sufficiently small, then f varies slowly, and the expansion is justified. The possibility of an emergent small parameter, such as E_m, allows us to use a perturbation approach.

Of course, it is not difficult to extend these results to deal with cases having an additional small perturbation. For example, the flat sides of the stadium might be given some small wriggles. An example of this type that has been published [41] makes the radius at one end of the stadium billiard slightly larger than at the other end, so that the stadium flat sides are not quite parallel. Note that the periodic orbits change their mathematical character drastically if the stadium sides are tilted. The periodic orbit, trace formula approach to these problems is rather difficult [39].

Another case, Fig. 10, which has been experimentally realized [42] is the ice cream cone billiard, which is a unit circle for $|\theta| > \beta$, and a triangular shaped region for $|\theta| < \beta$. This turns out to have essentially the same effective potential as for the stadium with $x \to \theta - \pi/2$ and $[-a, a] \to [\pi - \beta, \beta]$.

In these cases, our method turns out to be essentially identical to the results based on the standard Born-Oppenheimer ansatz used earlier [38,40]. For the ice cream cone billiard,

the ansatz is a generalization of the customary one. Alternatively, the full wave function we predict, $\Psi(x,y)$, approximates the Born-Oppenheimer wave function.

Another large class of states to which our method applies is the region near stable period orbits. The tractability of this case has been recognized in previous work of course. For example, the Hamiltonian might be expanded near the stable orbit and then be quantized.

IX. COMPARISON WITH PREVIOUS WORK ON COMBINED QUASICLASSICAL AND PERTURBATION THEORY

A. General theory

Combined QCA and PT has been used and studied extensively. One general scheme is to find approximations for the classical invariant tori, either analytically or numerically, and then use EBK methods to quantize them.

An important method is that of the Birkhoff-Gustavson normal form [6,33]. This is closely related to the Darling-Denison form [43]. In this case, the usual starting point is two or more harmonic-oscillator Hamiltonians coupled by terms consisting of polynomials in the displacements and momenta. Successive canonical transformations eliminate the coupling terms, order by order, in favor of a normal form Hamiltonian, which basically consists of products of powers of the original harmonic oscillators.

The result is complicated in the sense that computer algebra is needed to find the normal form, and in addition, a Padé technique is employed [44]. Indeed, the result gives an approximation to *all*, or some large class of invariant tori, not just the intersection with the SS, in *one single formula*. The entire torus is needed to do standard EBK quantization.

Since we find the invariant loops as a by-product of our approach, it can be regarded as a variation of this already existing work. However, we remark that solving the square billiard in the presence of a flux line could not be done by Birkhoff-Gustavson, because there *is* no classical perturbation. Our method gives localized states which are similar to new, quantized, resonant tori, but these new tori do not exist at the classical level.

Also, our method gives much simpler results. We need to find only the invariant loops in the surface of section, and not the whole torus to which an invariant loop corresponds. We do not need to integrate around the torus from one intersection with the SS to the next. Rather, knowledge of the phase $\omega(E)$ enables us to complete the quantization. Thus our method has the usual advantages of using the surface of section. Not only that, we have *different formulas* for *different regions* of angular momenta. Thus, our results are comparatively simple to derive and write down, since it breaks up the problem into small pieces, rather than finding one complicated formula describing everything at once.

Although discussion of the convergence of the series is given in the context of Birkhoff-Gustavson [33], mention is not made of the relationship of the small parameter ϵ to $\hbar$. Since the same formula is used to describe all the resonances, which have different scales, etc., that discussion would be quite difficult.

A related though more numerical approach is to propagate numerically an orbit which lies on an invariant torus [31]. Fourier methods then allow a determination of the fundamental frequencies, the ω_i's, and then the torus itself. This method is quite efficient.

Although the SS technique is often used to display results, it is seldom used as a calculational tool. The one exception we have found is the work of Noid and Marcus [45]. Indeed, they used two surfaces of section, SS1 and SS2, which cut the invariant tori in topologically different ways. By numerical propagation of an orbit, they found the invariant loops l_1 and l_2 on these two SS. EBK quantization makes the area of the loops take the form $area_1=\hbar(n_1+\frac{1}{2})$, $area_2=\hbar(n_2+\frac{1}{2})$, assuming the usual $\frac{1}{2}$ for librational states. In two dimensions, the SS, and thus in effect the orbits, depend on two parameters, one of which is taken as the energy. These parameters are varied until the areas of the loops have the above relation to integers. Noid and Marcus obtained numerically good results for the energy levels. Use of SS's gave some advantage computationally. The method does not rely on a small parameter, but only on the existence of sufficiently simple invariant loops.

A different kind of effort [7,11,12,29,30] studies the transition away from the Berry-Tabor trace formula, valid for completely integrable systems at $\epsilon=0$, and towards the Gutzwiller trace formula, usually considered only for hard chaos. We will show the relation of our work to that in a future submission.

There are also rigorous mathematical results in this field [5]. We are not sure how to make a simple statement of what has been rigorously proved. It seems that Lazutkin has succeeded in quantizing the KAM set at the EBK level, and has shown that the number of levels found is correct. The wave functions are also correct, up to the possibility discussed below of accidental degeneracy.

B. Recent applications

In addition to the subject of magnetic effects on square billiards [34] and deformed rectangular billiards [36], there are a number of papers [15–18] in the recent literature which have uncovered interesting phenomena in this perturbative case. Indeed, it was this literature which led us to develop this theory. In these papers, the basic object is the weakly deformed circular billiard described by the function $\Delta R(\theta)$ as in Fig. 3. The older literature [35] treats such boundary perturbations in more ordinary quantum perturbation theory, valid only for $k\sqrt{\epsilon}\ll 1$.

Some of this work has practical applications in the construction of tiny lasers with almost circular resonant cavities [15]. The case $\Delta R=\cos 2\theta$ [15,18] has some special features in that a nearly circular elliptical billiard has this as its leading-order term, and such a billiard is integrable. Higher-order corrections are quite interesting since they are responsible for island chains, etc. That is, this case can be thought of as an integrable elliptical billiard with leading perturbation proportional to ϵ^2. We reserve a more detailed analysis to future publication, however.

Other work is more concerned with localization theory [22]. Non-KAM cases were studied which are classically diffusive, i.e., technically the systems show hard chaos. The existence of a rather complete and analytic solution for large $k\epsilon$ was not suspected, so statistical studies of localization

and energy level statistics were undertaken with supporting heuristic arguments. In particular, the nearly circular Bunimovich stadium billiard [16,19,20] was studied.

It is rigorously known [16] that because of the lack of a second derivative, the long-time behavior is diffusive in angular momentum space and there are no invariant tori at all. A somewhat related case is to take ΔR as a ''random'' finite Fourier series constrained to have M maxima and minima [17]. There is a third, large, parameter M, whose relationship with k and ϵ must be specified. The possibility of large higher derivatives of the perturbation can modify or invalidate KAM theory.

The general interest in the quantum version of these non-KAM cases has been to find changes of wave-function localization behavior as a function of the parameters, and to interpret these regimes as belonging to different types of dynamical localization. At some large enough k, the large derivatives in these cases start to play a role, and stationary phase arguments and perturbation theory breaks down. The contribution of our paper is for values of k less than this characteristic magnitude. A detailed study of this breakdown is underway.

However, *to some order* in $\sqrt{\epsilon}$, for the resonances, approximate invariant tori continue to exist, and we can find them and quantize them. If these tori break down at order r, then for $k\epsilon^{r/2} \ll 1$, the quantum system acts as if the tori exist. For larger k, quantum mechanics can ''see'' the dissolution of the tori, and our theory no longer applies. This transition and its interpretation is the theme of several papers [16,17,19,20].

This can also be interpreted as the fact that quantum mechanics and its limiting classical mechanics may have quite different long-time behavior. The two approximately coincide up to a time $\tau_\hbar$, a time usually interpreted as $\hbar/\Delta$, where Δ is the mean level spacing of the quantum levels which are appreciable in the initial wave packet. Beyond this time, the classical behavior is irrelevant to the understanding of the quantum behavior. Thus, the long-time classical behavior, which is the focus of most classical theory, can be irrelevant unless it also happens to describe the short-time behavior, $t < \tau_\hbar$. Quantum mechanics does not ''care'' if approximate invariant tori are indeed mathematically invariant. It only cares that the classical orbit stays close enough to a torus for long enough that phase interference effects can establish the quantum states.

An explanation [19,20] used in this context is ''cantorus,'' which is a Cantor set invariant phase-space structure replacing an invariant KAM torus which has disappeared as a parameter is changed. Cantori [28] can ''trap'' for a while classical orbits, thus influencing the short-time behavior. The *invariance* of a cantorus is what makes it interesting and difficult in the classical context, but this invariance is of no relevance for quantum systems. Insofar as cantori ''explain'' the short-time behavior, they explain some of the quantum integrability even in the absence of classical integrability. However, in our opinion, the existence of approximate invariant loops is a better and more complete explanation.

X. FUTURE WORK

Several avenues for future work are underway. It will be interesting to see how the perturbation approach breaks down in some detail, in cases such as the stadium billiard. Working out and verifying higher-order effects in detail is of some interest. It would be good to extend the method to arbitrary surfaces of section rather than the special ones we have used. A number of interesting problems are raised by cases such as $\Delta R = \cos 2\theta$. Other perturbation systems are of interest, for example, coupled quartic oscillators [29]. It is no doubt possible to extend the method to weakly coupled harmonic oscillators. A general theory of whispering gallery orbits in sufficiently smooth convex billiards [39,37] might be possible, but would require a better T operator than we have used above, since diffraction effects become important for short classical skips between nearby boundary points. Extension to $d > 2$ may be in principle possible but in practice is likely to be difficult. There are interesting numerical results, for example in ray splitting billiards [37], that suggest it might be possible to solve the case that *two* or more orbits contribute to a specific T operator, requiring a sort of matrix version of our theory. Finally, it could be hoped to use this technique to study states in pseudointegrable systems, which have nonisolated orbits that, however, do not lie on a simple torus in phase space. Many of these projects are underway [24].

XI. SUMMARY

Weakly perturbed integral Hamiltonian systems in two dimensions have been much studied, both classically and quasiclassically. Up to now, the methods employed, although effective numerically, did not lend themselves to visualization and simple formulas. By using surface of section techniques as generalized to quantum systems by Bogomolny, we obtain results very much like familiar results from WKB theory, at least in the leading orders.

The relation of the perturbation parameter ϵ, to the dimensionless wave number k, or equivalently to $1/\hbar$, is crucial. The size of $k\epsilon$, the number of wavelengths in the change of a billiard boundary, is not especially important, for resonant states. Rather, the main parameter is $k\sqrt{\epsilon}$. Results are quite simple if $k\epsilon^{3/2}$ is small. In that case, the derivatives of the perturbation do not enter directly into the solution. This also means, of course, that for given small ϵ, there can be quite large effects proportional to $\sqrt{\epsilon}$. For example, it is not enough that an optical resonator have ''optically flat'' sides, if it is to approximate well a perfect square resonator. For nonresonant states, the size of $k\epsilon$ determines whether ordinary perturbation methods are valid or not.

A first description of what we have done is that we have solved Eq. (5) in WKB and stationary phase approximation for the wave functions and energy levels as a power series in $\sqrt{\epsilon}$, or in ϵ, depending on the phase-space region. We have a criterion for whether the series should be $\sqrt{\epsilon}$ or ϵ. To be a bit more accurate, we find explicit solutions for the leading $\sqrt{\epsilon}$ and first few powers, and provide a procedure which rapidly becomes tedious, for the higher orders. In this paper we concentrated on the leading order or two, and not on issues of the breakdown of the series. The latter is related to the degree of smoothness of the perturbation.

A second description is in terms of quantum perturbation theory. We do a quantum perturbation theory about an initial state $\psi = \exp(il\theta)$. However, unless $\sqrt{\epsilon}/\hbar \ll 1$, this is not an

ordinary perturbation theory, but rather a *degenerate* perturbation theory, in which of order $\sqrt{\epsilon}/\hbar$ unperturbed states are mixed, in the resonant case, or $\epsilon/\hbar$ states in the KAM set case. This can be seen in Fig. 2, when it is realized that unperturbed states are represented by action values spaced $\hbar$ apart. So, we can effectively carry out the diagonalization of the matrix mixing these states in this case. The fact that a form of degenerate perturbation theory is needed was apparently first pointed out by Shuryak [46], in the case of resonances. Degenerate perturbation theory is usually a matter of diagonalizing matrices, but the quasiclassical approximation lets us avoid that.

A third interpretation of what we do, at least in some cases, is that we are able to find explicit approximate formulas for the invariant loops. In other cases, we find adiabatic invariants. These loops have the topology of circles and are invariant under the surface of section map. The loops are found as a power series in ϵ for tori in the KAM set. We find explicit formulas for the new, resonant, invariant loops as a power series in $\sqrt{\epsilon}$. We find explicit formulas for the separatrices in this way also. The loops can be quantized, in essence by EBK theory, and the formulation then gives the other quantization of the motion away from the SS. We also have expressions for the scale of the perturbation around a resonant torus, I_{pq}. Having these formulas, we can find the quasiclassical quantization conditions and the wave functions.

A fourth interpretation is that we have a generalization of adiabatic approximations like the Born-Oppenheimer method. This happens because we have constructed an ordering of contributions to the phases according to how fast these phases vary. It is also similar to multiple time-scale analysis.

We, however, do *not* find any stochasticity, or any secondary resonance phenomena, since these are smaller than any power of ϵ. This has the unfortunate implication that our work does not shed much light on the quantization of mixed chaos. It would indeed be surprising, however, if the quantized wave functions of extended stochastic regions had simple analytic expressions. In fact, mixed chaos consists of three regions, a KAM-like region with invariant tori, a region presumably strongly chaotic and treatable by the GTF and its improvements, and a transition region between the two. It is the transition region which has resisted attempts to quantize it.

Our results are simple and familiar on the one hand, but rather complex on the other. That is, we obtain results of a type familiar from standard WKB theory. But, the results are complex in the sense that the results of classical KAM theory are complex. Various cases are possible depending on the choice of system. Since the quantum theory must follow the classical up to a certain level of approximation, depending on $\hbar$, the quantum results must also be complex in this sense.

ACKNOWLEDGMENTS

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APPENDIX: $\epsilon^2 \ll \hbar \leq \epsilon^{3/2}$

We indicate here how to find the resonant solution if $kb^3 \simeq 1$, $kb^2 \gg 1$ and $kb^4 \ll 1$. We expand S_0 about the nominal stationary point Θ_{pq} to third order:

$$S_0(\theta-\theta')/\hbar + l_{pq}(\theta'-\theta) \simeq S_0(\Theta_{pq})/\hbar + l_{pq}\Theta_{pq} - k\frac{d}{4}\delta\theta^2 + \frac{1}{8}kdd_1\delta\theta^3. \tag{A1}$$

The order ϵ term is

$$S_2(\theta,\theta') \simeq b^2[dV_2(\theta) + dV_2'(\theta)\delta\theta] \tag{A2}$$

and the order ϵ^2 term is

$$S_4(\theta,\theta') = db^4 V_4(\theta). \tag{A3}$$

We have normalized to the second derivative of S_0 which is written in terms of the constant d. Note that V_2' is not the derivative of V_2.

The wave function is

$$\psi = \exp(-g)\exp\, ikd(bf_1 + b^2 f_2 + b^3 f_3 + b^4 f_4). \tag{A4}$$

We expand

$$f_1(\theta') = f_1(\theta+\Theta_{pq}) + \delta\theta f_1'(\theta+\Theta_{pq}) + \delta\theta^2 f_1''(\theta+\Theta_{pq}),$$
$$f_2(\theta') = f_2(\theta+\Theta_{pq}) + \delta\theta f_2'(\theta+\Theta_{pq}), \tag{A5}$$
$$f_3(\theta') = f_3(\theta+\Theta_{pq}).$$

We know f_1 and the part of f_2 which has no Fourier components which are multiples of q. We investigate the conditions on the solution which arise at order kb^3.

The stationary phase point of the integral satisfies the equation

$$-\frac{1}{2}\delta\theta + bf_1' + 24d_1\delta\theta^2 + bf_1''\delta\theta + b^2 f_2' + b^2 V_2' = 0.$$

The last four terms are of order b^2, while the first two are of order b, since the leading expression for $\delta\theta$ is of order b. Thus, we find

$$\delta\theta = 2bf_1' + 2b^2[24d_1(f_1')^2 + 2f_1'f_1'' + f_2' + V_2'].$$

The integration is performed, yielding a condition at order b^3

$$d_1 f_1'^3 + 2f_1'^2 f_1'' + 2f_1'(f_2' + V_2') = f_3(\theta) - f_3(\theta+\Theta_{pq}) + \text{const.} \tag{A6}$$

Taking the q average of both sides, we have an equation determining $\overline{f_2}$, the part of f_2 invariant under q average,

$$\overline{f}'_{2q} = -\overline{V}'_{2q} - f'_1 f''_1 - \frac{4}{3} d_1 (f'_1)^2 + \frac{c}{f'_1}. \tag{A7}$$

The part of f_3 vanishing under q average is determined from

$$f_3(\theta) - f_3(\theta + \Theta_{pq}) = 2 f'_1 (f'_2 - \overline{f}'_2 + V'_2 - \overline{V}'_{2q})$$

and $\overline{f}_3$ is determined by the next-order condition. Note that if f_1 is double-valued, so is f_3. It can therefore change the shape of the resonance islands.

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